Decomposition of Treatment Effect with Interference Accounting for Network Change

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Motivation

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- Classical program evaluation methods are based on independence between indivduals (Rubin (1974)).
- Causal effect is defined by potential outcomes:

$$Y(1) \quad - \quad Y(0)$$

$Y(1)$ ↑ Potential outcome when the individual is treated

$Y(0)$ ↑ Potential outcome when the individual is untreated

- Only one of potential outcome is observed, and the other is counterfactual.

Motivation: An Example (J. Cai, Janvry, and Sadoulet (2015))

- Outcome (Y): Indicator for whether farmer buy the weather insurance.
- Treatment (D): Attending an information session about the benefits of the insurance.
- Estimating the treatment (causal) effect using a regression:

$$Y_i = \beta_0 + \beta_I D_i + \varepsilon_i.$$

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- When the treatment is randomly assigned, and individuals are independent:

Potential buying if the individual does not attend the session ↓

$$\beta_I = E[Y(1) - Y(0)]$$

↑ Potential buying if the individual attends the session:

- Estimate: $\hat{\beta}_I = 0.14$.

Motivation: An Example (J. Cai, Janvry, and Sadoulet (2015))

- Outcome (Y): Indicator for whether farmer buy the weather insurance.
- Treatment (D): Attending an information session about the benefits of the insurance.
- Network Link (A_{ij}): Indicator for whether individuals i and j are friends.
- The authors estimate the *social network effect* using the following regression model:

$$Y_i = \beta_0 + \beta_I D_i + \beta_T \underbrace{\sum_{j \neq i} (A_{ij}/5) D_j}_{\text{Fraction (number) of treated friends}} + \epsilon_i$$

Fraction (number) of treated friends

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$$Y_i = \beta_0 + \beta_I D_i + \beta_T \underbrace{\sum_{j \neq i} (A_{ij}/5) D_j}_{\uparrow} + \epsilon_i$$

Fraction (number) of treated friends

- $\hat{\beta}_I = 0.029$, $\hat{\beta}_T = 0.291$: implies having one more treated friend will increase $29\%p/5 = 5.8\%p$.
- Causal interpretation of $\hat{\beta}_T$? $\rightarrow$ depends on the structure of the potential outcome.

Motivation: Potential Outcome and Causal Effects When Indivdiuals Interact

- When individuals interact, potential outcomes are generally determined by:
  - **(i)** The treatment status of all individuals $\boldsymbol{d} = (d_1, ..., d_N)$;
  - **(ii)** The underlying network structure, represented by the $N \times N$ adjacency matrix $\boldsymbol{A} = [A_{ij}]_{ij}$.

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  - **(ii)** The underlying network structure, represented by the $N \times N$ adjacency matrix $\boldsymbol{A} = [A_{ij}]_{ij}$.
- Causal effects of changes in treatment status from $\boldsymbol{d}'$ to $\boldsymbol{d}$, given (exogenous) network $\boldsymbol{A}$:

$$Y(\boldsymbol{d}, \boldsymbol{A}) \quad - \quad Y(\boldsymbol{d}', \boldsymbol{A})$$

$Y(\boldsymbol{d}, \boldsymbol{A})$ ↑ Potential outcome when treatment status is given by $\boldsymbol{d}$

$Y(\boldsymbol{d}', \boldsymbol{A})$ ↑ Potential outcome when treatment status is given by $\boldsymbol{d}'$

Notes:

- ▷ $\boldsymbol{d}, \boldsymbol{d}' \in \{0,1\}^N$ are vectors representing treatment statuses for all individuals.
- ▷ $\boldsymbol{A}$ is a $N \times N$ binary matrix with $A_{ij} = 1$ if i, j are linked, $A_{ii} = 0$ for all i.

Motivation: Evidence of Network Change

- Network is usually assumed to be fixed, or exogenous (i.e., not affected by the treatment).
  - Assuming fixed (predetermined) network will be valid in the short run.
- Some empirical evidence that network is also affected by treatment:
  - Offering free savings account makes households less dependent (Dupas, Keats, and Robinson (2019)), but increases probability of forming links (Comola and Prina (2021))
  - Introducing micro finance can reduce network density (total links/possible links) (Banerjee et al. (2024))
  - Job search assistance reduces information sharing (Caria, Franklin, and Witte (2023))
  - Increasing in empathy levels can affect friendship network in classroom, resulting in reducing bullying behaviors (Hu (2023))

Motivation: Decomposition of Causal Effects

- What if the network is also influenced by the treatment?
- Let $\boldsymbol{A}(\boldsymbol{d})$ be potential network when the treatment is $\boldsymbol{d}$.
- Causal changes in outcome through changes in treatment status from $\boldsymbol{d}'$ to $\boldsymbol{d}$

$$Y(\boldsymbol{d}, \boldsymbol{A}(\boldsymbol{d})) \quad - \quad Y(\boldsymbol{d}', \boldsymbol{A}(\boldsymbol{d}'))$$

↑ Potential outcome when treatment status is given by $\boldsymbol{d}$

↑ Potential outcome when treatment status is given by $\boldsymbol{d}'$

Notes:

- ▷ $\boldsymbol{d}, \boldsymbol{d}' \in \{0,1\}^N$ are vectors representing treatment statuses for all individuals.
- ▷ $\boldsymbol{A}(\boldsymbol{d})$ is a $N \times N$ potential network adjacency matrix when treatment status is given by $\boldsymbol{d}$.

Motivation: Decomposition of Causal Effects

- What if the network is also influenced by the treatment? $\Rightarrow$ result in two distinct causal effects.
- Let $\boldsymbol{A}(\boldsymbol{d})$ be potential network when the treatment is $\boldsymbol{d}$.
- Causal changes in outcome through changes in treatment status from $\boldsymbol{d}'$ to $\boldsymbol{d}$

$$Y(\boldsymbol{d}, \boldsymbol{A}(\boldsymbol{d})) - Y(\boldsymbol{d}, \boldsymbol{A}(\boldsymbol{d}')) \quad + \quad Y(\boldsymbol{d}, \boldsymbol{A}(\boldsymbol{d}')) - Y(\boldsymbol{d}', \boldsymbol{A}(\boldsymbol{d}'))$$

↑ Causal effect from changes in network ($\boldsymbol{A}(\boldsymbol{d}') \to \boldsymbol{A}(\boldsymbol{d})$) given treatment status $\boldsymbol{d}$

↑ Causal effect from changes in treatment ($\boldsymbol{d}' \to \boldsymbol{d}$), given network structure $A(\boldsymbol{d}')$

Notes:

- $\boldsymbol{d}, \boldsymbol{d}' \in \{0,1\}^N$ are vectors representing treatment statuses for all individuals.
- $\boldsymbol{A}(\boldsymbol{d})$ is a $N \times N$ potential network adjacency matrix when treatment status is given by $\boldsymbol{d}$.

Motivation: Research Question and Key Findings

- **Research Question**
  - How can we identify causal effects and the decomposition if the network is influenced by treatment (as well as endogenous)?
- **Main Findings**
  - Assumptions:
    - Dyadic (potential) link formation.
    - Linear response function for potential outcomes.
  - Identification of causal effects and decomposition:
    - In the case of exogenous treatment (randomized experiment).
    - In the case of parallel-trend and no-anticipation assumptions (quasi-experiment).
  - Decomposition allows us to understand a mechanism of the program.

Related Literature: Identification of causal effects with interference

- **Randomized Experiments**
  J. Cai, Janvry, and Sadoulet (2015), Aronow and Samii (2017), Leung (2020), Forastiere, Airoldi, and Mealli (2021), Leung (2022), Vazquez-Bare (2022)
- **Quasi-Experiments**
  Xu (2023), Dall'erba et al. (2021), Butts (2021), Auerbach, Y. Cai, and Rafi (2024)
- **Double Randomization**
  Baird et al. (2018), Hudgens and Halloran (2008), Viviano (2019), DiTraglia et al. (2023)
- **Treatment effect accounting for network change**
  Comola and Prina (2021): explicitly investigated the treatment effects with network change Differences

Model

Model: Setup

- Assume there are G groups, each consisting of N individuals.
- Let t represent time periods when data is available from more than two periods.
- For simplicity, omit time $_t$ and group $_g$ subscripts when there is no confusion.
- Notations:
  - $Y_i \in \mathbb{R}$: Observed outcome.
  - $D_i \in \{0, 1\}$: Treatment indicator for individual i,
  - $\boldsymbol{D} = (D_1, ..., D_N)' \in \{0, 1\}^N$: Treatment vector for all individuals in the group.
  - $A_{ij} \in \{0, 1\}$: Observed network link that indicates whether individuals i and j are linked.
  - $\boldsymbol{A} = [A_{ij}]$: $N \times N$ adjacency matrix representing the network, with $[\boldsymbol{A}]_{ij} = A_{ij}$.

Model: Potential Responses for Links

Assumption 1 (Dyadic Response on Potential Links; DR)

Each pair (i, j)'s potential link is determined by (d_i, d_j) only, and mean-independent of the treatments of others, conditional on (D_i, D_j). Formally:

$$A_{ij}(\boldsymbol{d}) = A_{ij}(d_i, d_j), \quad w.p.1, \quad E[A_{ij}(d_i, d_j)|\boldsymbol{D}] = E[A_{ij}(d_i, d_j)|D_i, D_j], \quad \forall \boldsymbol{d} \in \{0, 1\}^N$$

- Example: Dyadic Network Formation Model with Homophily

$$A_{ij}(\boldsymbol{d}) = A_{ij}(d_i, d_j) = \mathbb{1}\{\theta_0 + \theta_1|d_i - d_j| + r_{ij} \geq 0\},$$

where r_{ij}: unobserved characteristic, independent of other pairs' treatment conditional on (D_i, D_j).

  - e.g.: Goldsmith-Pinkham and Imbens (2013), Graham (2017).

Model: Potential Responses for Outcomes

Assumption 2 (Linear Response on Potential Outcomes; LR)

For each individual i, the potential outcome is generated by the following response function:

$$Y_i(\boldsymbol{d}) = \beta_0 + \beta_I d_i + \beta_T Q_i(\boldsymbol{d}) + \beta_U R_i(\boldsymbol{d}) + \varepsilon_i(d_i),$$

where:

(i) $Q_i(\boldsymbol{d}) = \sum_{j \neq i} A_{ij}(d_i, d_j) d_j$ *is the # of treated neighbors,*

(ii) $R_i(\boldsymbol{d}) = \sum_{j \neq i} A_{ij}(d_i, d_j)(1 - d_j)$ *is the # of untreated neighbors,*

(iii) $\varepsilon_i(d_i)$ *is mean zero error term that does not have ATT, and* $E[\varepsilon_i(d_i)|\boldsymbol{D}] = E[\varepsilon_i(d_i)|D_i]$.

- $\tilde{Y}_i(d_i) := \beta_0 + \beta_I d_i + \varepsilon_i(d_i)$ is individual component of potential outcome.
- $\beta_T Q_i(\boldsymbol{d}) + \beta_U R_i(\boldsymbol{d})$ capture interaction.
  - β_T, β_U are spillover effects from treated, untreated neighbors.

Model: Potential Responses for Outcomes

- The corresponding observed outcome follows a linear network model:

$$Y_i = \beta_0 + \beta_I D_i + \beta_T Q_i + \beta_U R_i + \varepsilon_i,$$

where Q_i, R_i are observed # of treated, untreated neighbors.

  - In the example of J. Cai, Janvry, and Sadoulet (2015),
    - Y_i: Buying the insurance,
    - D_i: Attending an info. session about benefits of the insurance,
    - Q_i: The number of friends attended the info. session,
    - $\beta_U = 0$.

Model: Potential Responses for Outcomes

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  - By Leung (2020), the potential outcome is determined by $(d_i, Q_i(\boldsymbol{d}), R_i(\boldsymbol{d}))$ if and only if
    - **(i)** The network is anonymous: individuals cannot identify specific neighbors;
    - **(ii)** Interactions occur only with neighbors within distance 1 (local interference).
  - Assumption (LR) assumes additional linearity in the response function.

Model: Potential Responses for Outcomes

- The corresponding observed outcome follows a linear network model:

$$Y_i = \beta_0 + \beta_I D_i + \beta_T Q_i + \beta_U R_i + \varepsilon_i,$$

where Q_i, R_i are observed $\#$ of treated, untreated neighbors.

  - When both treatment $\boldsymbol{D}$ and network $\boldsymbol{A}$ are exogenous (i.e., $E[\varepsilon_i|\boldsymbol{D}, \boldsymbol{A}] = 0$), coefficients β can be recovered by least squares if $(1, D_i, Q_i, R_i)$ are linearly independent.
  - This setting allows correlation between potential links and potential outcomes (i.e., network can be endogenous).

Model: Causal Effects and Decomposition

- Direct effect is defined by the effect of **own treatment** (d_i), and decomposed by:

Direct Treatment Effect (DTE)

$$Y_i(\boldsymbol{d}) = \tilde{Y}_i(d_i) + \beta_T \sum_{j=1}^{N} A_{ij}(d_i, d_j)d_j + \beta_U \sum_{j=1}^{N} A_{ij}(d_i, d_j)(1 - d_j)$$

Direct Network Effect (DNE)

Note: $\tilde{Y}_i(d_i) = \beta_0 + \beta_I d_i + \varepsilon_i(d_i)$ is the individual component of potential outcome.

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Direct Network Effect (DNE)

  - Compare two situations: (i) only i is treated $(\boldsymbol{d}')$; (ii) no individual is treated $(\boldsymbol{d})$.

$$Y_i(\boldsymbol{d}') - Y_i(\boldsymbol{d}) = \underbrace{\tilde{Y}_i(1) - \tilde{Y}_i(0)}_{DTE=\beta_I+\varepsilon_i(d_i)} + \underbrace{\beta_U \sum_{j \neq i} [A_{ij}(1,0) - A_{ij}(0,0)]}_{DNE}.$$

Note: $\tilde{Y}_i(d_i) = \beta_0 + \beta_I d_i + \varepsilon_i(d_i)$ is the individual component of potential outcome.

Model: Causal Effects and Decomposition

- Indirect effect is defined by the effect of **one other individual's treatment** (d_j), and decomposed by:

Indirect Treatment Effect (ITE)

$$Y_i(\boldsymbol{d}) = \tilde{Y}_i(d_i) + \beta_T \sum_{j=1}^{N} A_{ij}(d_i, \boldsymbol{d}_j)\boldsymbol{d}_j + \beta_U \sum_{j=1}^{N} A_{ij}(d_i, \boldsymbol{d}_j)(1 - \boldsymbol{d}_j)$$

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Indirect Network Effect (INE)

  - Compare two situations: (i) only j is treated $(\boldsymbol{d}')$; (ii) no individual is treated $(\boldsymbol{d})$.

$$Y_i(\boldsymbol{d}') - Y_i(\boldsymbol{d}) = \beta_T A_{ij}(0,1) - \beta_U A_{ij}(0,0) = \underbrace{(\beta_T - \beta_U)A_{ij}(0,0)}_{ITE} + \underbrace{\beta_T(A_{ij}(0,1) - A_{ij}(0,0))}_{INE}$$

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Indirect Network Effect (INE)

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$$= \underbrace{(\beta_T - \beta_U)A_{ij}(0,1)}_{ITE} + \underbrace{\beta_U(A_{ij}(0,1) - A_{ij}(0,0))}_{INE}$$

Model: Causal Effects and Decomposition

- Average direct effects: compares $\boldsymbol{d}'$: only i is treated, $\boldsymbol{d}$: no individual is treated.

$$E[Y_i(\boldsymbol{d}') - Y_i(\boldsymbol{d})|\boldsymbol{D} = \boldsymbol{d}] = \underbrace{\beta_I}_{ADTE} \underbrace{+ \beta_U \sum_{j \neq i} E\left[A_{ij}(1,0) - A_{ij}(0,0)|D_i = 1, D_j = 0\right]}_{ADNE}.$$

- Average indirect effects: compares $\boldsymbol{d}'$: only j is treated, $\boldsymbol{d}$: no individual is treated.

$$E[Y_i(\boldsymbol{d}') - Y_i(\boldsymbol{d})|\boldsymbol{D} = \boldsymbol{d}] = \underbrace{(\beta_T - \beta_U)E[A_{ij}(0,0)|D_i = 0, D_j = 1]}_{AITE} + \underbrace{\beta_T E[A_{ij}(0,1) - A_{ij}(0,0)|D_i = 0, D_j = 1]}_{AINE}$$

Model: Parameters of Interest

- The parameters of interest $\boldsymbol{\pi} = (\pi^{DT}, \pi^{DN}, \pi^{IT}, \pi^{IN})'$:

$$\textbf{Average Direct Treatment Effect } (\pi^{DT}) = \beta_I,$$

$$\textbf{Average Direct Network Effect } (\pi^{DN}) = \beta_U (N-1) H(1,0), \qquad N\text{: \# individuals in groups},$$

$$\textbf{Average Indirect Treatment Effect } (\pi^{IT}) = (\beta_T - \beta_U) m(0,1),$$

$$\textbf{Average Indirect Network Effect } (\pi^{IN}) = \beta_T H(0,1),$$

where

$$\text{ATT on individual component:} \quad \beta_I := E[\tilde{Y}_i(1) - \tilde{Y}_i(0) | D_i = 1],$$

$$\text{ATT on link:} \quad H(d_i, d_j) := E[A_{ij}(d_i, d_j) - A_{ij}(0,0) | D_i = d_i, D_j = d_j],$$

$$\text{Conditional average of untreated link:} \quad m(d_i, d_j) := E[A_{ij}(0,0) | D_i = d_i, D_j = d_j].$$

Identification

Identification: Overview

Intuition of identifying the decomposition $\boldsymbol{\pi} = (\pi^{DT}, \pi^{DN}, \pi^{IT}, \pi^{IN})$:

Stage 1 Identify the distribution of potential links using dyadic data, which includes the observed links between individuals.

- Baseline expectation of link ($m(0,1)$),
- ATT of links ($H(1,0)$, $H(0,1)$),
- Predicted network ($E[A_{ij}|D_i, D_j]$).

Stage 2 The outcome coefficient β is identified by outcome regression using individual-level data. This step uses the predicted network from Stage 1.

- e.g., Kelejian and Piras (2014), König, Liu, and Zenou (2019), Lee et al. (2021)

Stage 3 Recover the decomposition $\boldsymbol{\pi}$ by parameters identified in the first two stages.

Identification: Two Experimental Designs

- Overall assumption on distribution:
  - **(a)** The individual-level and dyadic-level data are identically distributed, independent over groups.
  - **(b)** $\Pr(D_i = d, D_j = e) \in (0, 1)$ for all $(d, e) \in \{0, 1\}^2$.

- Consider two experimental designs:
  - Randomized experiment
    - Observe post-treatment information.
    - Exogeneity of treatment (allow endogeneity of network).
  - Quasi experiment
    - Observe pre- and post-treatment information.
    - Parallel trends and no-anticipation for both network links (A_{ijt}) and outcomes (Y_{it}).

Identification: Identification Under Randomized Experiment

Assumption 3 (Exogeneity; EX)

Treatment is exogenous: $E[\varepsilon_i(d_i)|D_i] = E[\varepsilon_i(d_i)] = 0$, $E[A_{ij}(d_i, d_j)|D_i, D_j] = E[A_{ij}(d_i, d_j)]$.

Identification: Identification Under Randomized Experiment

Assumption 4 (Exogeneity; EX)

Treatment is exogenous: $E[\varepsilon_i(d_i)|D_i] = E[\varepsilon_i(d_i)] = 0$, $E[A_{ij}(d_i, d_j)|D_i, D_j] = E[A_{ij}(d_i, d_j)]$.

Stage 1 Consider a saturated dyadic regression:

$$E[A_{ij}|D_i, D_j] = \zeta_1 + \zeta_2 D_i + \zeta_3 D_j + \zeta_4 D_i D_j =: \boldsymbol{W}'_{ij}\boldsymbol{\zeta},$$

where $\boldsymbol{W}_{ij} = (1, D_i, D_j, D_i D_j)'$. The coefficient ζ consists of

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} E[A_{ij}|D_i = 0, D_j = 0] \\ E[A_{ij}|D_i = 1, D_j = 0] - E[A_{ij}|D_i = 0, D_j = 0] \\ E[A_{ij}|D_i = 0, D_j = 1] - E[A_{ij}|D_i = 0, D_j = 0] \end{pmatrix} = \begin{pmatrix} M(0,0) \\ H(1,0) \\ H(0,1) \end{pmatrix},$$

which is identified when $\Pr(D_i \neq D_j) > 0$.

Identification: Identification Under Randomized Experiment

Assumption 5 (Exogeneity; EX)

Treatment is exogenous: $E[\varepsilon_i(d_i)|D_i] = E[\varepsilon_i(d_i)] = 0$, $E[A_{ij}(d_i, d_j)|D_i, D_j] = E[A_{ij}(d_i, d_j)]$.

Stage 2 Consider an a regression on observed outcome:

$$\begin{aligned} E[Y|\boldsymbol{D}] &= \beta_0 + \beta_I D_i + \beta_T \sum_{j\neq i} E[A_{ij}|D_i, D_j]D_j + \beta_U \sum_{j\neq i} E[A_{ij}|D_i, D_j](1-D_j). \\ &= \beta_0 + \beta_I D_i + \beta_T \sum_{j\neq i} (\boldsymbol{W}'_{ij}\boldsymbol{\zeta})D_j + \beta_U \sum_{j\neq i} (\boldsymbol{W}'_{ij}\boldsymbol{\zeta})(1-D_j) \\ &=: \boldsymbol{Z}_i(\boldsymbol{\zeta})'\boldsymbol{\beta}, \end{aligned}$$

where $\boldsymbol{Z}_i(\boldsymbol{\zeta}) = \left(1, D_i, \sum_{j\neq i}(\boldsymbol{W}'_{ij}\boldsymbol{\zeta})D_j, \sum_{j\neq i}(\boldsymbol{W}'_{ij}\boldsymbol{\zeta})(1-D_j)\right)'$ and $\boldsymbol{\beta} = (\beta_0, \beta_I, \beta_T, \beta_U)$ that is identified when $E[\boldsymbol{Z}_i(\boldsymbol{\zeta})\boldsymbol{Z}_i(\boldsymbol{\zeta})']$ is nonsingular.

Identification: Identification Under Randomized Experiment

Assumption 6 (Exogeneity; EX)

Treatment is exogenous: $E[\varepsilon_i(d_i)|D_i] = E[\varepsilon_i(d_i)] = 0$, $E[A_{ij}(d_i, d_j)|D_i, D_j] = E[A_{ij}(d_i, d_j)]$.

Stage 3 Recover $\boldsymbol{\pi}$ by definition:

$$\boldsymbol{\pi} = \begin{pmatrix} \beta_I \\ (N-1)\beta_U H(1,0) \\ (\beta_T - \beta_U)m(0,1) \\ \beta_T H(0,1) \end{pmatrix} = \begin{pmatrix} \beta_I \\ (N-1)\beta_U \zeta_2 \\ (\beta_T - \beta_U)\zeta_1 \\ \beta_T \zeta_3 \end{pmatrix}.$$

Identification: Identification Under Randomized Experiment

Proposition 1 (Identification Under Randomized Experiment)

Under Assumptions (DR), (LR), and (EX), we have

1. $\boldsymbol{\zeta}$ *is identified by the dyadic regression* $E[A_{ij}|D_i, D_j] = \boldsymbol{W}'_{ij}\boldsymbol{\zeta}$*;*
2. $\boldsymbol{\beta} = (\beta_0, \beta_I, \beta_T, \beta_U)'$ *is identitifed by the outcome regression:*

$$E[Y_i|\boldsymbol{D}] = \beta_0 + \beta_I D_i + \beta_T \sum_{j \neq i} (\boldsymbol{W}'_{ij}\boldsymbol{\zeta}) D_j + \beta_U \sum_{j \neq i} (\boldsymbol{W}'_{ij}\boldsymbol{\zeta})(1 - D_j) := \boldsymbol{Z}_i(\boldsymbol{\zeta})'\boldsymbol{\beta};$$

3. *Decomposition is identified by* $\boldsymbol{\pi} = \left(\beta_I, \quad (N-1)\beta_U\zeta_2, \quad (\beta_T - \beta_U)\zeta_1, \quad \beta_T\zeta_3\right)'$.

Identification: Identification Under Quasi-Experiment with Parallel Trend

- We observe both pre-treatment ($t = 0$) and post-treatment ($t = 1$) information.
- Denote $\beta_{It}, \beta_{Tt}, \beta_{Ut}$ be outcome coefficients at period t, and Δ be first-difference operator.

Assumption 7 (No Anticipation; NA)

There is no-anticipation on individual component, potential links, and potential outcome:
For $d \in \{0,1\}, (d_i, d_j) \in \{0,1\}^2$, *(i)* $\varepsilon_{i0} = \varepsilon_{i0}(d)$ *a.s.; (ii)* $A_{ij0} = A_{ij0}(d_i, d_j)$ *a.s.; (iii)* $\beta_{I0} = 0$, $\beta_{T0} = \beta_{U0}$.

Assumption 8 (Parallel Trend; PT)

Paralell trend holds for individual component, potential links, and potential outcome:
(i) $E[\Delta\varepsilon_i(0)|D_i] = E[\Delta\varepsilon_i(0)]$*; (ii)* $E[\Delta A_{ij}(0,0)|D_i, D_j] = E[\Delta A_{ij}(0,0)]$*; (iii)* $\beta_{U0} = \beta_{U1}$.

Identification: Identification Under Quasi-Experiment with Parallel Trend

Stage 1

- Again, consider the following staturated regressions:

$$E[A_{ijt}|D_i, D_j] = \zeta_{1t} + \zeta_{2t}D_i + \zeta_{3t}D_j + \zeta_{4t}D_iD_j =: \boldsymbol{W}'_{ij}\boldsymbol{\zeta}_t,$$

- Next, define $\boldsymbol{\xi} := \boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_0$. Then, $\boldsymbol{\xi}$ is the difference-in-differences coefficient that consists of

$$\begin{pmatrix} \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} E[\Delta A_{ij}|D_i = 1, D_j = 0] - E[\Delta A_{ij}|D_i = 0, D_j = 0] \\ E[\Delta A_{ij}|D_i = 0, D_j = 1] - E[\Delta A_{ij}|D_i = 0, D_j = 0] \end{pmatrix} = \begin{pmatrix} H(1,0) \\ H(0,1) \end{pmatrix}.$$

- Then, $m(0,1) = E[A_{ij1}|D_i = 0, D_j = 1] - H(0,1) = \zeta_{31} - \xi_3 = \zeta_{30} + \zeta_{11}$.
- Coefficients $\boldsymbol{\zeta}_t$, $\boldsymbol{\xi}$ are identified when $\Pr(D_i \neq D_j) > 0$.

Identification: Identification Under Quasi-Experiment with Parallel Trend

Stage 2

- The first-differenced observed outcome is given by

$$\Delta Y_i = \Delta\beta_0 + \beta_I D_i + \beta_T Q_{i1} + \beta_U (R_{i1} - S_{i0}) + \Delta\varepsilon_i,$$

where Q_{i1}, R_{i1} are observed $\#$ of treated, untreated neighbors at $t = 1$, and S_{i0} is the $\#$ of neighbors at $t = 0$.

- Taking conditional expectation, we have the similar result:

$$E[\Delta Y_i | \boldsymbol{D}] = \boldsymbol{X}_i(\boldsymbol{\zeta})'\boldsymbol{\beta},$$

where $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_0)$,
$\boldsymbol{X}_i(\boldsymbol{\zeta}) = \left(1, D_i, \sum_{j \neq i} (\boldsymbol{W}'_{ij}\boldsymbol{\zeta}_1) D_j, \sum_{j \neq i} \left[(\boldsymbol{W}'_{ij}\boldsymbol{\zeta}_1)(1 - D_j) - (\boldsymbol{W}'_{ij}\boldsymbol{\zeta}_0)\right]\right)'$ and
$\boldsymbol{\beta} = (\Delta\beta_0, \beta_{I1}, \beta_{T1}, \beta_{U1})$.

- Coefficient $\boldsymbol{\beta}$ is identified when $E[\boldsymbol{X}_i(\boldsymbol{\zeta})\boldsymbol{X}_i(\boldsymbol{\zeta})']$ is nonsingular.

Identification: Identification Under Quasi-Experiment with Parallel Trend

Stage 3 Recover $\boldsymbol{\pi}$ by definition:

$$\boldsymbol{\pi} = \begin{pmatrix} \beta_I \\ (N-1)\beta_U H(1,0) \\ (\beta_T - \beta_U) m(0,1) \\ \beta_T H(0,1) \end{pmatrix} = \begin{pmatrix} \beta_{I1} \\ (N-1)\beta_{U1}\xi_2 \\ (\beta_{T1} - \beta_{U1})(\zeta_{30} + \zeta_{11}) \\ \beta_{T1}\xi_3 \end{pmatrix}.$$

Identification: Identification Under Quasi-Experiment with Parallel Trend

Proposition 2 (Identification With Parallel Trend)

Under Assumptions (DR), (LR), (NA), and (PT), we have

1. ζ_0, ζ_1 *are identified by the dyadic regressions* $E[A_{ijt}|D_i, D_j] = \boldsymbol{W}'_{ij}\boldsymbol{\zeta}_t$, *and* $\boldsymbol{\xi} = \boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_0$.
2. $\boldsymbol{\beta} = (\Delta\beta_0, \beta_{I1}, \beta_{T1}, \beta_{U1})'$ *is identitifed by the outcome regression:*

$$E[\Delta Y_i|\boldsymbol{D}] = \Delta\beta_0 + \beta_{I1}D_i + \beta_{T1}\sum_{j\neq i}(\boldsymbol{W}'_{ij}\boldsymbol{\zeta}_1)D_j + \beta_{U1}\sum_{j\neq i}\left\{(\boldsymbol{W}'_{ij}\boldsymbol{\zeta}_1)(1-D_j) - (\boldsymbol{W}'_{ij}\boldsymbol{\zeta}_0)\right\}$$

$$:= \boldsymbol{X}_i(\boldsymbol{\zeta})'\boldsymbol{\beta}.$$

3. *Decomposition* $\boldsymbol{\pi}$ *is identified by* $\boldsymbol{\pi} = \Big(\beta_{I1}, \quad (N-1)\beta_{U1}\xi_2, \quad (\beta_{T1}-\beta_{U1})(\zeta_{30}+\zeta_{11}), \quad \beta_{T1}\xi_3\Big)$.

Identification: Remark

1. Regardless of the experimental design, for the decomposition, we need identification of:
   - **(i)** ATT for network links ($H(\cdot,\cdot)$) (since $M(0,0)$ is directly observed).
   - **(ii)** ATT for individual component (β_I),
   - **(iii)** Outcome coefficients (β_T, β_U).

   Thus, the approach could be applied to another specific experimental designs, e.g., double randomization (Hudgens and Halloran (2008)).
2. The definition of casual parameters and decomposition is from the linearity of potential outcome.
3. The main assumptions (DR) and (LR) can be relaxed with more algebra.

Estimation and Inference

Estimation and Inference: Estimation Using Data From Randomized Experiment

- Coefficient ζ is estimated by

$$\hat{\zeta} = \left(\sum_{g=1}^{G} \sum_{i \neq j} \boldsymbol{W}_{ijg} \boldsymbol{W}'_{ijg} \right)^{-1} \sum_{g=1}^{G} \sum_{i \neq j} \boldsymbol{W}_{ijg} A_{ijg}.$$

- Coefficient β is estimated by

$$\hat{\boldsymbol{\beta}} = \left(\sum_{g=1}^{G} \sum_{i=1}^{N} \boldsymbol{Z}_{ig}(\hat{\boldsymbol{\zeta}}) \boldsymbol{Z}_{ig}(\hat{\boldsymbol{\zeta}})' \right)^{-1} \sum_{g=1}^{G} \sum_{i=1}^{N} \boldsymbol{Z}_{ig}(\hat{\boldsymbol{\zeta}}) Y_{ig}.$$

- Decomposition $\hat{\boldsymbol{\pi}}$ is estimated by

$$\hat{\boldsymbol{\pi}} = \left(\hat{\beta}_I, \quad (N-1)\hat{\beta}_U \hat{\zeta}_2, \quad (\hat{\beta}_T - \hat{\beta}_U)\hat{\zeta}_1, \quad \hat{\beta}_T \hat{\zeta}_3 \right).$$

Notes:

$$\boldsymbol{W}_{ijg} = \begin{pmatrix} 1 \\ D_{ig} \\ D_{jg} \\ D_{ig} D_{jg} \end{pmatrix}$$

$$\boldsymbol{Z}_{ig}(\boldsymbol{\zeta}) = \begin{pmatrix} 1 \\ D_{ig} \\ \sum_{j \neq i} \hat{A}_{ijg} D_{jg} \\ \sum_{j \neq i} \hat{A}_{ijg}(1 - D_{jg}) \end{pmatrix}$$

$$\hat{A}_{ijg} = \boldsymbol{W}'_{ijg} \boldsymbol{\zeta}$$

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_I \\ \beta_T \\ \beta_U \end{pmatrix}$$

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} M(0,0) \\ H(1,0) \\ H(0,1) \end{pmatrix}$$

Estimation and Inference: Estimation Using Data From Quasi-Experiment With Parallel Trend

- Coefficient ζ_t is estimated by

$$\hat{\zeta}_t = \left(\sum_{g=1}^{G}\sum_{i\neq j} \boldsymbol{W}_{ijg}\boldsymbol{W}'_{ijg}\right)^{-1}\sum_{g=1}^{G}\sum_{i\neq j}\boldsymbol{W}_{ijg}A_{ijtg}.$$

- The difference-in-differences coefficient is $\boldsymbol{\xi} = \boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_0$.
- Coefficient $\boldsymbol{\beta}$ is estimated by

$$\hat{\boldsymbol{\beta}} = \left(\sum_{g=1}^{G}\sum_{i=1}^{N} \boldsymbol{X}_{ig}(\hat{\boldsymbol{\zeta}})\boldsymbol{X}_{ig}(\hat{\boldsymbol{\zeta}})'\right)^{-1}\sum_{g=1}^{G}\sum_{i=1}^{N}\boldsymbol{X}_{ig}(\hat{\boldsymbol{\zeta}})\Delta Y_{ig}.$$

- Decomposition $\hat{\boldsymbol{\pi}}$ is estimated by

$$\hat{\boldsymbol{\pi}} = \left(\hat{\beta}_{I1},\quad (N-1)\hat{\beta}_{U1}\hat{\xi}_2,\quad (\hat{\beta}_{T1}-\hat{\beta}_{U1})[\hat{\boldsymbol{\zeta}}_1]_1,\quad \hat{\beta}_{T1}\hat{\xi}_3\right).$$

Notes:

$$\boldsymbol{W}_{ijg} = \begin{pmatrix} 1 \\ D_{ig} \\ D_{jg} \\ D_{ig}D_{jg} \end{pmatrix}$$

$$\boldsymbol{X}_{ig}(\boldsymbol{\zeta}) = \begin{pmatrix} 1 \\ D_{ig} \\ \sum_{j\neq i} \hat{A}_{ij1g}D_{jg} \\ \sum_{j\neq i}\begin{bmatrix} \hat{A}_{ij1g}(1-D_{jg}) \\ -\hat{A}_{ij0g} \end{bmatrix} \end{pmatrix}$$

$$\hat{A}_{ijtg} = \boldsymbol{W}'_{ijg}\boldsymbol{\zeta}_t$$

$$\boldsymbol{\zeta} = (\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1)$$

$$\boldsymbol{\beta} = \begin{pmatrix} \Delta\beta_0 \\ \beta_{I1} \\ \beta_{T1} \\ \beta_{U1} \end{pmatrix}$$

$$\begin{pmatrix} \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} H(1,0) \\ H(0,1) \end{pmatrix}$$

$$[\boldsymbol{\zeta}_1]_1 = M(0,0)$$

Estimation and Inference: Inference

Proposition 3 (Asymptotic Properties I (Randomized Experiment))

Let $(\zeta^\star, \beta^\star, \pi^\star)$ be true values of parameters. Suppose Assumptions (LR), (DR), (EX) hold, in addition assume (i) $E[Y_{ig}^4] < \infty$; (ii) $\boldsymbol{R_W} := E[\boldsymbol{W}_{ijg}\boldsymbol{W}'_{ijg}]$ is nonsingular; (ii) $\boldsymbol{R_Z} := E[\boldsymbol{Z}_{ig}(\zeta^\star)\boldsymbol{Z}_{ig}(\zeta^\star)']$ is nonsingular. Then, as $G \to \infty$, $(\hat{\zeta}, \hat{\beta}, \hat{\pi})$ is consistent, and asymptotically normal:

$$\hat{V}_{\zeta}^{-1/2}\sqrt{G}(\hat{\zeta} - \zeta^\star) \xrightarrow{d} N(0,1),$$
$$\hat{V}_{\beta}^{-1/2}\sqrt{G}(\hat{\beta} - \beta^\star) \xrightarrow{d} N(0,1),$$
$$\hat{V}_{\pi}^{-1/2}\sqrt{G}(\hat{\pi} - \pi^\star) \xrightarrow{d} N(0,1),$$

where $\hat{V}_p$ is the plug-in clustered standard error based on the empirical influence functions for $p \in \{\zeta, \beta, \pi\}$. Influence Functions

Estimation and Inference: Remark

1. When all identifying assumptions hold conditioning on covariates, we can apply the propensity score reweighting method proposed by Abadie (2005) for identification and estimation.
   - ▷ For random vectors X_i and V_i, we have $E[X_i|D_i = 1, V_i] = E\left[\frac{D_i}{\Pr(D_i=1|V_i)} X_i \middle| V_i\right]$.

2. In sparse networks, we can allow large N for the limiting distribution and use individual and dyadic variation. Asymptotic Theory Under Bounded Degree
   - ▷ E.g., Stein (1972), Chen, Goldstein, and Shao (2010), Ross (2011), Leung (2020).
   - ▷ Specifically, if the maximum degree $(\max_i \sum_j A_{ij})$ is $o_p(N)$, and $E[Y_{ig}^6] < \infty$ then we have the same limiting distribution when $N, G \to \infty$.

Simulation

Simulation: Design 1

- Design 1: Randomized Experiment

$$\text{Treatment Assignments:} \quad D_i \sim \text{Binomial}(1, P_D), \quad P_D = 0.5,$$

$$\text{Potential/Observed Links:} \quad A_{ij}(d_i, d_j) = \mathbb{1}\left\{\mathcal{I}_{\boldsymbol{\theta}}(d_i, d_j) \geq \nu_{ij}\right\}, \quad A_{ij} = A_{ij}(D_i, D_j), \quad \boldsymbol{\theta} = (-1, 0.1, 0.1, 1)',$$

$$\mathcal{I}_{\boldsymbol{\theta}}(d_i, d_j) := \theta_1 + \theta_2 d_i + \theta_3 d_j + \theta_4 d_i d_j,$$

$$\text{Observed Outcome:} \quad Y_i = \beta_0 + \beta_I D_i + \beta_T \sum_{j \neq i} A_{ij} D_j + \beta_U \sum_{j \neq i} A_{ij}(1 - D_j) + \left(u_i + \sum \nu_{ij}\right),$$

where $\nu_{ij} \sim N(0,1)$, $u_i \sim N(0,1)$, $\boldsymbol{\beta} = (2, 1, 0.8, 0.6)$.

- By consruction, (EX) holds and the underlying network is endogenous.

Simulation: Design 2

- Design 2: Quasi-experiments with parallel trend + no-anticipation

$$\text{Treatment Assignments:} \quad D_i \sim \text{Binomial}(1, P_D), \quad P_D = 0.5,$$

$$\text{Potential/Observed Links:} \quad A_{ij1}(d_i, d_j) = \mathbb{1}\{\mathcal{I}_{\boldsymbol{\theta}}(d_i, d_j) + h_1(D_i, D_j) \geq \nu_{ij1}\}, \quad A_{ij1} = A_{ij1}(D_i, D_j), \quad \boldsymbol{\theta} = (-1, 0.1, 0.1, 1)',$$

$$A_{ij0} = \mathbb{1}\{h_0(D_i, D_j) \geq \nu_{ij0}\}$$

$$\text{Observed Outcome:} \quad Y_{i1} = \beta_{01} + \beta_{I1} D_i + \beta_{T1} \sum_{j \neq i} A_{ij1} D_j + \beta_{U1} \sum_{j \neq i} A_{ij1}(1 - D_j) + \left(u_{i1} + \sum \nu_{ij1}\right),$$

$$Y_{i0} = \beta_{00} + \beta_{U1} \sum_{j \neq i} A_{ij0} + \left(u_{i0} + \sum \nu_{ij0}\right),$$

where $h_0(d_i, d_j) = \mathcal{I}_{\boldsymbol{\omega}}(d_i, d_j)$ with $\boldsymbol{\omega} = (-1.5, 0.3, 0.3, -1)'$, $h_1(d_i, d_j) = h_0(d_i, d_j) - \mathcal{I}_{\boldsymbol{\theta}}(0, 0)$, $\nu_{ijt}, u_{it} \sim N(0, 1)$, and $\boldsymbol{\beta}_1 = (2, 1, 0.8, 0.6)$, $\boldsymbol{\beta}_0 = (1, 0, 0.6, 0.6)$.

- By consruction, (NA), (PT) holds and the underlying network is endogenous.

Simulation: Root Mean Squared Errors

Table 1: Simulation: Median of Estimates for Decomposition

	Design 1				Design 2			
G	π^{DT}	π^{DN}	π^{IT}	π^{IN}	π^{DT}	π^{DN}	π^{IT}	π^{IN}
50	1.0011	0.2597	0.0312	0.0197	1.0052	0.1794	0.0136	0.0148
100	1.0026	0.2725	0.032	0.0199	1.0045	0.2116	0.0129	0.0156
200	0.9998	0.2814	0.0311	0.0202	0.9969	0.2214	0.0132	0.016
400	0.9995	0.2851	0.0319	0.0203	1.0058	0.226	0.0136	0.0163
800	1.0033	0.2862	0.032	0.0203	0.9992	0.2324	0.0132	0.0164
TRUE	1	0.29	0.032	0.02	1	0.235	0.013	0.016

Notes: Number of individuals is $N = 20$, and number of simulations is $B = 10,000$. This table shows the mean over all replication: $\frac{1}{B}\sum_{b=1}^{B}\hat{\pi}^{x}$, $x \in \{DT, DN, IT, IN\}$.

Simulation: Root Mean Squared Errors

Table 2: Simulation: MSE of Estimates for Decomposition

	Design 1				Design 2			
G	π^{DT}	π^{DN}	π^{IT}	π^{IN}	π^{DT}	π^{DN}	π^{IT}	π^{IN}
50	0.6166	0.0903	0.0027	0.0001	1.6011	0.2159	0.002	0.0002
100	0.3012	0.0413	0.0014	0.0001	0.7761	0.0908	0.001	0.0001
200	0.1463	0.0192	0.0007	0	0.384	0.0406	0.0005	0
400	0.073	0.0092	0.0003	0	0.1869	0.0187	0.0002	0
800	0.0358	0.0046	0.0002	0	0.0897	0.0089	0.0001	0

Notes: Number of individuals is $N = 20$, and number of simulations is $B = 10,000$. This table shows the mean squared error over all replication: $\frac{1}{B}\sum_{b=1}^{B}(\hat{\pi}^x - (\pi^x)^\star)^2$, $x \in \{DT, DN, IT, IN\}$.

Simulation: Root Mean Squared Errors

Table 3: Simulation: 95% Coverage Rate of Estimates for Decomposition

	Design 1				Design 2			
G	π^{DT}	π^{DN}	π^{IT}	π^{IN}	π^{DT}	π^{DN}	π^{IT}	π^{IN}
50	0.9363	0.9042	0.9311	0.932	0.9436	0.9335	0.9342	0.8982
100	0.9394	0.9154	0.9365	0.9391	0.9429	0.9243	0.9342	0.9209
200	0.9433	0.9334	0.9443	0.9433	0.943	0.9346	0.9429	0.9304
400	0.9428	0.9383	0.9469	0.9415	0.9443	0.941	0.9465	0.9381
800	0.9492	0.9437	0.9506	0.9487	0.9506	0.9441	0.9487	0.9441

Notes: Number of individuals is $N = 20$, and number of simulations is $B = 10,000$. This table shows the 95% coverage rate over all replication: $\frac{1}{B}\sum_{b=1}^{B} \mathbb{1}\{(\pi^x)^\star \in [\hat{\pi}^x \pm 1.96\text{se}(\hat{\pi}^x)]\}$, $x \in \{DT, DN, IT, IN\}$.

Simulation: 95% Coverage Rates

Table 4: Simulation: Bias Assessment

	Design 1			Design 2		
G	$\hat{\beta}$	$\hat{\beta}^E$	$\hat{\beta}^N$	$\hat{\beta}$	$\hat{\beta}^E$	$\hat{\beta}^N$
50	0.4821	1.7681	2.7246	0.6544	0.9526	0.8959
100	0.3399	1.7658	2.7249	0.4603	0.9406	0.8959
200	0.2377	1.7655	2.7251	0.3242	0.9323	0.8959
400	0.1665	1.765	2.7257	0.2259	0.9267	0.8964
800	0.1181	1.7651	2.7258	0.1581	0.9232	0.8966

Notes: Number of individuals is $N = 20$, and number of simulations is $B = 10,000$. This table shows the overall mean absolute error (MAE) $\frac{1}{KB}\sum_{b=1}^{B}\sum_{k=1}^{K}|\hat{\beta}_k - \beta_k^{\star}|$. $\hat{\beta}^E$ represent the coefficient in the regression of Y_i on $(1, D_i, Q_i, R_i)$ for Design 1, and that of ΔY_i on $(1, D_i, Q_{i1}, R_{i1} - S_{i0})$ for Design 2. $\hat{\beta}^N$ is the coefficient in the regression of outcomes on $(1, D_i)$ only.

Empirical Illustration

Empirical Illustration: Data and Variables

- Comola and Prina (2021) (CP, hereafter) investigate the impact of providing savings account to households to consumption, by a randomized experiment conducted in Nepal (2009–2011).
- The data consist of 915 households across 19 villages.
- Findings in CP:
  - Positive direct and indirect effects on meat consumption.
  - 0.002%p increase in the probability of forming financial links.
- CP estimate:

$$Y_{i1} = \beta_1 \sum_{j \neq i} \tilde{A}_{ij0} Y_{j1} + \beta_2 \sum_{j \neq i} \Delta \tilde{A}_{ij} Y_{j1} + \gamma D_i + \delta_1 \sum_{j \neq i} \tilde{A}_{ij0} D_j + \delta_2 \sum_{j \neq i} \Delta \tilde{A}_{ij} D_j + \varepsilon_{i1},$$

with $E[\varepsilon_{i1} | \boldsymbol{A}_1, \boldsymbol{A}_0, \boldsymbol{D}] = 0$. And compute $\partial E[Y_{i1} | \boldsymbol{D}] / \partial \boldsymbol{D}'$ for direct, indirect effects.

Empirical Illustration: Average Treatment Effects on Treated of Links

Table 5: Average Treatment Effects on Treated of Links

Var	ΔA^s	ΔA^s	ΔA
Constant	-0.0009	-0.0009	-0.0031
	(0.0012)	(0.0012)	(0.0024)
Some Treated	0.0021		
	(0.0016)		
D_i		0.0021	0.0039*
		(0.002)	(0.0023)
D_j		0.0023	0.0039*
		(0.0018)	(0.0023)
$D_i \times D_j$		-0.0025	-0.0034
		(0.003)	(0.0034)
Observations		56,308	

Notes: The dependent variable in the third column is $A_{ij1} - A_{ij0}$, while in the first two columns, it is $A^s_{ij1} - A^s_{ij0}$, where $A^s_{ijt} = A_{ijt} / \sum_{j \neq i} A_{ijt}$ represents the row-normalized links. Standard errors are reported in parentheses. *,**,*** denote the significance levels at 10%, 5%, and 1%, respectively.

Empirical Illustration: Decomposition of Treatment Effects

Table 6: Decomposition of Treatment Effects

	M1		M2		M3		M3	
	Direct	Indirect	Direct	Indirect	Direct	Indirect	Direct	Indirect
Treatment	240.7**	185***	207.4**	215.4***	211.8***	138.4***	275.2***	195.6***
	(115.8)	(38.3)	(99.3)	(44.8)	(54.7)	(28.9)	(52.3)	(41.2)
Network	-169.9	1.6	-202.4	1.6	-135	0.8	-188.7	1.2
	(385.7)	(3.6)	(459.9)	(3.7)	(306.8)	(1.8)	(428.8)	(2.8)
Total	70.9	186.6***	4.9	217.1***	76.8	139.2***	86.5	196.8***
	(380.1)	(36.4)	(445)	(43)	(292.7)	(28.1)	(415.1)	(39.9)
Obs.	915		915		915		612	
R^2	0.40		0.58		0.98		0.99	

Notes: The dependent variable in the first two columns (M1, M2) is Y_1. In model (M2) a dummy variable $\mathbb{1}\{Y_1 = 0\}$ is used to control individuals who do not consume meat. In model (M3) the dependent variable is $\log(Y_1)$ for $Y_1 > 0$, set to zero for $Y_1 = 0$, and controls a dummy variable $\mathbb{1}\{Y_1 = 0\}$. In model (M4) the dependent variable is $\log(Y_1)$ and drop the observations with $Y_1 = 0$. The coefficients in columns (M3) and (M4) are adjusted by multiplying by the mean of Y_1 (1,057.43) to allow for comparison with the first two columns. Village fixed effects are included to account for variations in meat consumption across different villages.

Conclusion

Conclusion

- Identifying the decomposition of causal effects, accounting for the causal network changes.
- The decomposition helps in understanding the mechanism behind the program.
- The proposed methods consider two different experimental designs
  - Randomized experiment
  - Quasi-experiment with parallel trend
  - Other experimental designs may also be applicable.
- Future directions
  - Consider more flexible functional form of potential outcome to avoid risk of misspecification,
  - Consider endogenous peer effects.

Appendix: Major Differences with Comola and Prina (2021)

- The main differences of this study from Comola and Prina (2021) (CP, hereafter) are summarized as follows:
  - I propose clear causal interpretation using potential outcome framework, but CP's estimate has causal interpretation only under randomized experiment
  - I derive limiting distribution for inference
  - In CP, network change is time-varying, while I consider causal network change.
  - I propose decomposition of the causal effect which is not considered in previous works.

Back

Appendix: Influence Functions (IF)

Let $\mathcal{V}_g = \{(D_{ig}, Y_{ig}), (A_{ijg}, \boldsymbol{W}_{ijg}) : \forall i, \forall (i,j)\}$ be group-level data. The influence functions of $\boldsymbol{\zeta}$, $\boldsymbol{\beta}$ are:

$$\psi_{\boldsymbol{\zeta}}(\mathcal{V}_g, \boldsymbol{\zeta}) := \boldsymbol{B}_{\boldsymbol{W}}^{-1} \frac{1}{N(N-1)} \sum_{(i,j):i \neq j} \boldsymbol{W}_{ijg}(A_{ijg} - \boldsymbol{W}'_{ijg}\boldsymbol{\zeta}),$$

$$\psi_{\boldsymbol{\beta}}(\mathcal{V}_g, \boldsymbol{\zeta}, \boldsymbol{\beta}) := \boldsymbol{B}_{\boldsymbol{Z}}^{-1} \left[\boldsymbol{Z}_{ig}(\boldsymbol{\zeta})(Y_{ig} - \boldsymbol{Z}_{ig}(\boldsymbol{\zeta})'\boldsymbol{\beta}) - \boldsymbol{C}_{\boldsymbol{\zeta}}\psi_{\boldsymbol{\zeta}}(\mathcal{V}_g, \boldsymbol{\zeta})\right],$$

where $\boldsymbol{C}_{\boldsymbol{\zeta}} := E[\boldsymbol{Z}_{ig}(\boldsymbol{\zeta}^\star)\nabla_{\boldsymbol{\zeta}}(\boldsymbol{Z}_{ig}(\boldsymbol{\zeta}^\star)'\boldsymbol{\beta}^\star)]$. And the influence function $\psi_{\boldsymbol{\pi}}(\mathcal{V}_g, \boldsymbol{\zeta}, \boldsymbol{\beta})$ of $\boldsymbol{\pi}$ is given by:

$$\begin{pmatrix} \psi_{\boldsymbol{\beta},2}(\mathcal{V}_g, \boldsymbol{\zeta}, \boldsymbol{\beta}) \\ (N-1)\psi_{\boldsymbol{\beta},4}(\mathcal{V}_g, \boldsymbol{\zeta}, \boldsymbol{\beta})\zeta_2^\star + \beta_4^\star \psi_{\boldsymbol{\zeta},2}(\mathcal{V}_g, \boldsymbol{\zeta}) \\ \left(\psi_{\boldsymbol{\beta},3}(\mathcal{V}_g, \boldsymbol{\zeta}, \boldsymbol{\beta}) - \psi_{\boldsymbol{\beta},4}(\mathcal{V}_g, \boldsymbol{\zeta}, \boldsymbol{\beta})\right)\zeta_1^\star + (\beta_T^\star - \beta_U^\star)\psi_{\boldsymbol{\zeta},1}(\mathcal{V}_g, \boldsymbol{\zeta}) \\ \psi_{\boldsymbol{\beta},3}(\mathcal{V}_g, \boldsymbol{\zeta}, \boldsymbol{\beta})\zeta_3^\star + \beta_3^\star \psi_{\boldsymbol{\zeta},3}(\mathcal{W}_g, \boldsymbol{\zeta}) \end{pmatrix},$$

where $\psi_{\boldsymbol{b},k}(\cdot)$ denote k-th element in vector $\psi_{\boldsymbol{b}}(\cdot)$ for $\boldsymbol{b} \in \{\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\pi}\}$. Lastly, $\hat{V}_{\boldsymbol{\zeta}}$, $\hat{V}_{\boldsymbol{\beta}}$, $\hat{V}_{\boldsymbol{\pi}}$ are sample variance-covariance matrices of $\psi_{\boldsymbol{\zeta}}(\mathcal{V}_g, \hat{\boldsymbol{\zeta}})$, $\psi_{\boldsymbol{\beta}}(\mathcal{V}_g, \hat{\boldsymbol{\zeta}}, \hat{\boldsymbol{\beta}})$, $\psi_{\boldsymbol{\pi}}(\mathcal{V}_g, \hat{\boldsymbol{\zeta}}, \hat{\boldsymbol{\beta}})$, respectively, and $V^{-1/2}$ denote a square root matrix of V^{-1}. Back

Appendix: Asymptotic Results with Dependent Data

Let $Deg^\star = \max_{1\le i\le N} \sum_{j\ne i} A_{ij}$ be the maximum degree, and V_i be a square-integrable random vector. Suppose $Deg^\star = o_p(N)$.

(i) If $E[\|V_i\|^2] < \infty$, then $\sum_{i=1}^N V_i \xrightarrow{p} E[V_i]$ by applying Chebychev's inequality.

(ii) If $E[\|V_i\|^3] < \infty$, then $\mathrm{Var}(V_i)^{-1/2} \sum_{i=1}^N (V_i - E[V_i]) \xrightarrow{d} N(0,1)$ by applying Stein's Bound.

Therefore, the require regularity condition is $E[Y_{ig}^6] < \infty$. Back